Supersymmetry and generalized calibrations

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A static minimal energy configuration of a super p-brane in a supersymmetric $(n+1)$-dimensional spacetime is shown to be a ''generalized calibrated'' submanifold. Calibrations in $\mathbb{E}^{(1,n)}$ and AdS_{n+1} are special cases. We present several M-brane examples. [S0556-2821(99)09418-7]

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I. INTRODUCTION

A super p-brane [1–3] is a charged p-dimensional object moving in superspace, with a charge equal to its tension, which we set to unity. If the superspace is a supersymmetric one, then the $(n+1)$-dimensional spacetime will admit a timelike Killing vector field k and the motion will conserve a corresponding energy E. The fact that the p-brane is charged means that it couples to a superspace $(p+1)$-form potential A. There are therefore two contributions to the energy density of a static brane, one proportional to the p-volume density, the other an ''electrostatic'' energy density. The p-volume density is $\sqrt{\det m}$ where m is the induced world-space p-metric. The electrostatic energy density is the world-space dual of the p-form induced by $i_k A$; let us call this Φ. Thus, as we shall confirm below, stable static bosonic solutions to the world-volume field equations (the ''brane-wave'' equations) minimize a potential energy functional of the form

$$E=\int d^p\sigma[\nu\sqrt{\det m}+\Phi], \tag{1}$$

where σ^i $(i=1,...,p)$ are the world-space coordinates and ν is a redshift factor; for static spacetimes, $\nu=\sqrt{-k^2}$. We shall call these static configurations ''minimal world spaces.'' For a vacuum superspace background the spacetime is $(n+1)$-dimensional Minkowski space $\mathbb{E}^{(1,n)}$. In this case $\Phi=0$ because the $(p+1)$-form A is purely fermionic. In addition, $\nu=1$, so in this case a minimal world space is a minimal surface in $\mathbb{E}^n$ in the usual sense. We see that super p-branes provide a natural framework for an extension of the notion of a minimal surface in $\mathbb{E}^n$ to minimal world spaces in more general Riemannian spaces.

Supersymmetry did not play a major role in the above discussion, and it might be deemed an unnecessary complication. However, a large class of minimal surfaces in $\mathbb{E}^n$ is composed of calibrated surfaces for which the calibrating p-form, or ''calibration'' [4], admits a spinorial construction [5]. Let $x^m=(x^0,x^I)$ be the Minkowski spacetime coordinates, with x^I $(I=1,...,n)$ being Cartesian coordinates on $\mathbb{E}^n$. Then, for $p=1,2 \bmod 4$, the p-form calibration takes the form

$$\varphi=dx^{I_1}\wedge\cdots\wedge dx^{I_p}\epsilon^T\Gamma_{0I_1\cdots I_p}\epsilon, \tag{2}$$

where ϵ is a constant real spinor[1] normalized so that

$$\epsilon^T\epsilon=1, \tag{3}$$

and $\Gamma_{m_1\cdots m_k}$ are antisymmetrized products of the Minkowski spacetime Dirac matrices. Many calibrations of this form have been shown [6] to calibrate supersymmetric intersecting brane configurations in M theory (for which ϵ is a real 32-component spinor of the 11-dimensional Lorentz group). Similar calibrations were earlier discussed in the context of Calabi-Yau compactifications of string theory [7]. The essential point is that, given a tangent p-plane ξ, we can write $\varphi|_\xi$ as

$$\varphi|_\xi=\sqrt{\det m}\,\epsilon^T\Gamma_\xi\epsilon, \tag{4}$$

where Γ_ξ is the matrix

$$\Gamma=\frac{1}{p!\sqrt{\det m}}\varepsilon^{i_1\cdots i_p}\partial_{i_1}x^{I_1}\cdots\partial_{i_p}x^{I_p}\Gamma_{0I_1\cdots I_p}, \tag{5}$$

evaluated at the point to which ξ is tangent. Given the above restriction on the values of p, we have

$$\Gamma^2=1. \tag{6}$$

It follows the $\varphi|_\xi\leq \mathrm{vol}_\xi$ for all ξ. Since φ is also closed, this means that it is a calibration. The ''contact set'' of the calibration is the set of p-planes for which this inequality is saturated. By the theorem of Harvey and Lawson, these are the tangent planes of a minimal surface [4].

We see from Eq. (4) that an alternative characterization of the contact set is that set of p-planes ξ for which

$$\Gamma_\xi\epsilon=\epsilon. \tag{7}$$

Because of Eq. (6), this equation is automatically satisfied for any *given* tangent p-plane ξ and, since $\mathrm{tr}\,\Gamma=0$, the solutions ϵ belong to a ''1/2 supersymmetric'' subspace of spinor space. However, this solution space generally varies as ξ varies over the contact set, so the solution space of the set is generally smaller. The connection of calibrations to supersymmetry observed in [6] is based on the fact that the matrix Γ is closely related to a matrix with similar properties ap-

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[1]The requirement that ϵ be real restricts the spacetime dimension and, hence, the space dimension n; the extent to which the constraints on p and n may be relaxed will be discussed below.

pearing in the κ-symmetry transformations of the world-volume fields of the super p-brane in its Lagrangian formulation. In fact, for time-independent world-volume fields in the gauge $x^0=t$ the latter matrix reduces to the one of Eq. (5), but in this context the condition (7) is just the condition for preservation of some fraction of supersymmetry; the total number of preserved supersymmetries is the dimension of the solution space for ϵ. Note that although the specific form of the calibrating p-form given above required restrictions on both p and n, the results can be extended without difficulty to all those pairs (p,n) for which there exists a super p-brane action. The possibilities were classified in [3] under the assumption that the only bosonic world-volume fields are scalars, and these are the cases that we have in mind here. The 11-dimensional supermembrane [2] is a simple example with $(p,n)=(2,10)$. Of course, many of the results to be obtained will apply to other branes such as D-branes and the $M5$-brane if one sets to zero any nonscalar world-volume fields. In fact, most discussions to date of the relevance of calibrations to branes have been in the context of the $M5$-brane.

Most previous discussions have also concentrated on the case of branes in a flat Minkowski background. In considering the extension to nonflat backgrounds it is convenient to distinguish between those for which the energy functional of a p-brane is still equal to the integral of the p-volume and those for which the more general formula (1) is required. In the former case, a static p-brane is again a a minimal surface and supersymmetric minimal p-surfaces are calibrated by p-form calibrations in much the same way as in the above discussion. The examples of [7] are in this category. As we shall show here, the recently discussed [8] ''lump'' solitons on the $M2$-brane in an M-monopole background provide examples of Kähler calibrations of this type, and there are a number of related Kähler calibrations that we discuss. If, on the other hand, a static brane of minimal energy is *not* a minimal surface, then it cannot be a calibrated surface in the sense of [4], but it may still be a calibrated surface in a more general sense because it may be calibrated by a form that is *not* closed; examples are provided by the ''anti–de Sitter'' (''AdS'') calibrations [9] which calibrate p-surfaces of minimum p-brane energy in anti–de Sitter backgrounds. The AdS examples suggest a definition of a ''generalized calibration,'' which we present here; we then show, by a straightforward extension of an argument in [9], that p-surfaces calibrated by such generalized calibrations are ''minimal world spaces.'' We discuss a number of examples provided by M-branes in 1/2-supersymmetric M-theory backgrounds.

Another purpose of this article is to show how the connection between calibrations and super p-branes follows directly from the supersymmetry algebra of Noether charges [10,11]. In particular, we shall see that all the properties of generalized calibrations arise naturally in this context. For this reason we shall begin with this analysis. Since the supersymmetry charges are integrals of functions on phase space, it is natural to start from the Hamiltonian formulation, the details of which can be found in [12–14]. Having defined the notion of a generalized calibration and proved the associated minimal world-space property of calibrated surfaces, we then move on to the explicit examples.

II. SUPERSYMMETRY ALGEBRA AND THE CALIBRATION BOUND

We begin by making precise the constructions alluded to in the opening paragraph. Let $Z^M=(x^m,\theta^\mu)$ be the superspace coordinates. The geometry of superspace will be assumed to be determined by the Lorentz frame one-forms $E^A=(E^a,E^\alpha)$ and the $(p+1)$-form potential A; we will assume throughout that the only nonvanishing component fields are bosonic. The superspace $(p+1)$-form A appears in the Lagrangian form of the action via its pullback to the world volume and the integrand can be written in the form $\dot{Z}^M\mathcal{A}_M$ where the overdot denotes differentiation with respect to an arbitrary world-volume time t; the coefficients $\mathcal{A}_M$ are the components of the one-form $\mathcal{A}$ obtained by contraction of A with the p tangent vectors spanning a tangent p-plane ξ to the p-dimensional world space, i.e., $\mathcal{A}=A|_\xi$. Now let $x^m=(x^0,x^I)$, where $k=\partial/\partial x^0$ is the timelike Killing vector field, and fix the time reparameterization invariance by choosing $x^0=t$. Then

$$\dot{Z}^M\mathcal{A}_M=\Phi+\dot{x}^I\mathcal{A}_I+\dot{\theta}^\mu\mathcal{A}_\mu, \tag{8}$$

where Φ is a superfield with the electrostatic potential density as its lowest component. As is customary, we use the same symbol to denote a superfield and its lowest component. Note that Φ is the world-space dual of the pullback of i_kA. The world-space metric is $m_{ij}=E_i^aE_j^b\eta_{ab}$, where $E_i{}^a=\partial_iZ^ME_M{}^a$ and η is the Minkowski metric. Omitting fermions it reduces, in the $x^0=t$ gauge, to

$$m_{ij}=\partial_ix^I\partial_jx^Jg_{IJ}, \tag{9}$$

where g_{IJ} is the metric on a spatial section of spacetime; i.e., the p-metric on world space is induced from the n-metric on space.

The super p-brane Lagrangian density can be written as

$$\mathcal{L}=\dot{L}^MP_M-s^iE_i{}^a\tilde{p}_a-\frac{1}{2}\lambda(\tilde{p}^2+\det m), \tag{10}$$

where s^i and λ are Lagrange multipliers, and

$$\tilde{p}_a=E_a{}^M(P_M+\mathcal{A}_M). \tag{11}$$

This is not its fully canonical form because κ symmetry implies a constraint for which there is as yet no corresponding Lagrangian multiplier. However, this constraint affects only the fermionic variables. If we set all world-volume fermions to zero and use the identity $\partial_ix^m\mathcal{A}_m\equiv0$, then

$$\mathcal{L}=\dot{x}^mp_m-s^i\partial_ix^mp_m-\frac{1}{2}\lambda(\tilde{p}^2+\det m), \tag{12}$$

where, now $\tilde{p}^2=g^{mn}\tilde{p}_m\tilde{p}_m$, with $\tilde{p}_m=p_m+\mathcal{A}_m$ and g^{mn} the inverse of the spacetime metric $g_{mn}=E_m{}^aE_n{}^b\eta_{ab}$. Now set

$$p_m = (-\mathcal{H}, P_I) \tag{13}$$

and impose the $x^0 = t$ gauge, to get

$$\mathcal{L} = \dot{x}^I p_I - \mathcal{H} - s^i \partial_i x^I p_I, \tag{14}$$

where the Hamiltonian density $\mathcal{H}$ is found by solving the Hamiltonian constraint $\tilde{p}^2 + \det m = 0$. This yields

$$\mathcal{H} = \Phi + v^I \tilde{p}_I + \nu [g^{IJ}_{(n)} \tilde{p}_I \tilde{p}_J + \det m]^{1/2}, \tag{15}$$

where

$$v^I = \frac{g^{tI}}{(g^{tt})}, \quad \nu = 1\sqrt{-g^{tt}}, \tag{16}$$

and $g^{IJ}_{(n)}$ is the inverse of the spatial n-metric g_{IJ}. For simplicity, we now restrict ourselves to static spacetimes, for which v^I vanishes. In this case $\nu = \sqrt{-k^2}$. For static world-volume configurations we may set $\tilde{p}_I = 0$ and the Hamiltonian density reduces to

$$\mathcal{H} = \Phi + \nu \sqrt{\det m}. \tag{17}$$

Because $k = \partial_t$ is a Killing vector field, m_{ij} and Φ are time independent. The integral of $\mathcal{H}$ over the world space yields the potential energy functional E of Eq. (1).

Supersymmetries of the background are associated with Killing spinor fields. In many simple cases these take the form of a function times a *constant* spinor ϵ subject to a constraint of the form $\Gamma_* \epsilon = \epsilon$, where the constant matrix Γ_* is characteristic of the background. In particular, all 1/2-supersymmetric backgrounds have Killing spinors of this form with Γ_* a traceless matrix such that $\Gamma_*^2 = 1$. We shall assume here that the background superspace is of this type. In addition, we continue to make the inessential, but simplifying, assumptions that $p = 1,2 \, mod \, 4$, and that spinors are real. In these cases we may take the charge conjugation matrix to equal $\Gamma^{\underline{0}}$, where the underlining indicates a constant Lorentz frame matrix. The matrix Γ of Eq. (5) can be generalized to nonflat spacetimes by taking

$$\sqrt{\det m}\, \Gamma = \Gamma_{\underline{0}} \gamma, \tag{18}$$

where

$$\gamma = \frac{1}{p!} \varepsilon^{i_1 \cdots i_p} \partial_{i_1} x^{I_1} \cdots \partial_{i_p} x^{I_p} \Gamma_{I_1 \cdots I_p}. \tag{19}$$

Note that, given the restriction on the values of p, we have

$$\gamma^2 = -(-1)^p \det m. \tag{20}$$

Given our assumptions about the background, the supersymmetry charges will take the form of spinor functionals Q_α subject to the constraint $Q \Gamma_* = Q$. We will proceed by taking this projection to be implicit and checking consistency at the end. With this understood we can write the supersymmetry anticommutator as

$$\{Q, Q\} = \Gamma^{\underline{0}} \int d^p \sigma \;\; \nu [\Gamma^a \tilde{p}_a + \gamma]. \tag{21}$$

This result has not been directly established in full generality, but it agrees with the flat space case [10] and those nonflat cases that have been analyzed [11]. That it must be correct follows indirectly from considerations of κ symmetry, as will now be explained. The κ-symmetry transformations were given in the required form in [13]. In particular, the transformations of the fields Z^M are such that $\delta_\kappa Z^M E_M{}^a = 0$ and

$$\delta_\kappa Z^M E_M{}^\alpha = [\Gamma^a \tilde{p}_a - (-1)^p \gamma] \kappa. \tag{22}$$

Since $E_m{}^\alpha$ vanishes for a bosonic background and $E_\mu{}^\alpha = \delta_\mu{}^\alpha$ at $\theta^\mu = 0$, we may write the κ-symmetry transformation as

$$\delta_\kappa \theta = [\Gamma^a \tilde{p}_a - (-1)^p \gamma] \kappa. \tag{23}$$

This gauge transformation allows half the world-volume fermions to be gauged away. The supersymmetry transformations of the remaining half take the form

$$\delta_\epsilon \theta = \epsilon + \delta_{\kappa(\epsilon)} \theta, \tag{24}$$

where $\kappa(\epsilon)$ is now a specific matrix function of world-volume fields acting on the constant spinor ϵ. Its precise form depends on the κ-symmetry gauge choice, and it has the effect that $\delta_\epsilon \theta = M \epsilon$ for some matrix M [15]. The matrix M depends on the world-volume fields and hence, for static fields, on the world-space coordinates. At any given point on world space it is a matrix for which half the eigenvalues vanish. This is due to the fact that, locally, the brane realizes half the supersymmetry nonlinearly. The spinors ϵ corresponding to the linearly realized supersymmetries are the eigenspinors of M with zero eigenvalue. These are the supersymmetries preserved, locally, by the brane. The supersymmetries that are globally preserved correspond to those spinors ϵ that are eigenspinors of M with vanishing eigenvalue for every point on world space. Thus a world-volume configuration is supersymmetric if the equation $M(\sigma)\epsilon = 0$ has solutions for constant ϵ. This is equivalent to the equation $\delta_\epsilon \theta = 0$.

We now note the identity

$$[\Gamma^a \tilde{p}_a + \gamma] \delta_\kappa \theta = [(\tilde{p}^2 + \det m) - 2(\partial_i x^I p_I) \gamma^i \gamma] \kappa, \tag{25}$$

where $\gamma^i = m^{ij} \partial_j x^I \Gamma_I$. The right-hand side vanishes as a result of the Hamiltonian and world-space diffeomorphism constraints. Using these constraints, we deduce that $\delta_\epsilon \theta = 0$ implies

$$[\Gamma^a \tilde{p}_a + \gamma] \epsilon = 0, \tag{26}$$

which is an alternative form of the condition for preservation of supersymmetry that obviates the need to compute the matrix M. This is the Hamiltonian form of the supersymmetry condition derived in [16,17] from the Lagrangian formulation. Observe that the matrix multiplying ϵ is precisely the matrix appearing in the $\{Q,Q\}$ anticommutator, as required for consistency because it is manifest from Eq. (21) that eigenspinors of this matrix with vanishing eigenvalue correspond to supersymmetries that are preserved by the p-brane configuration.

We now turn to the consequences of the algebra (21) for static configurations with $\tilde{p}_I = 0$. Given the normalized spinor ϵ of the earlier discussion, we deduce from the positivity of $\{Q,Q\}$ that

$$-\tilde{p}_0 - \nu \epsilon^T \Gamma_{\underline{0}} \gamma \epsilon \geq 0. \tag{27}$$

The first term on the left-hand side is just $\nu\sqrt{\det m}$, by the Hamiltonian constraint. For the second term we have

$$\nu \epsilon^T \Gamma_{\underline{0}} \gamma \epsilon = \varphi|_\xi , \tag{28}$$

where $\varphi|_\xi$ is the evaluation on a tangent p-plane ξ of a p-form φ that is formally identical to Eq. (2), but now with $\Gamma_0 = \nu \Gamma_{\underline{0}}$, and Dirac matrices Γ_I satisfying $\{\Gamma_I, \Gamma_J\} = 2g_{IJ}$. Thus Eq. (27) is equivalent to

$$\varphi|_\xi \leq \nu \, \mathrm{vol}_\xi . \tag{29}$$

Setting $\tilde{p}_I = 0$, it follows from Eq. (21) that

$$2(Q\epsilon)^2 = \int d^p\sigma [\mathcal{H} - (i_k A + \varphi)|_\xi]. \tag{30}$$

The term in the integrand other than $\mathcal{H}$ constitute a central extension of the supertranslation algebra, and it follows from general considerations that such extensions are topological. In other words, the p-form $i_k A + \varphi$ must be closed. Equivalently,

$$d\varphi = \mathcal{F}, \tag{31}$$

where $\mathcal{F} = -d i_k A$. A p-form φ that satisfies both the inequality (29) and the condition (31) will be called a ‘‘generalized calibration.’’ This definition is modeled on the definition of an ‘‘AdS’’ calibration in [9], and AdS calibrations are examples of ‘‘generalized’’ calibrations; we will later provide some further examples that are not AdS calibrations.

It will now be shown that the calibrated manifolds of a generalization calibration are minimal world spaces; the argument is a straightforward extension of the one given for AdS calibrations in [9]. Let U be an open subset of a calibrated manifold of a generalized calibration φ. Then

$$\int_U d^p\sigma \, \nu \sqrt{\det m} = \int_U \varphi . \tag{32}$$

Now let V be another open p-dimensional set in the same homology class as U with $\partial V = \partial U$, and let D be a $(p+1)$-surface with $\partial D = U - V$. Then

$$\int_U \varphi = \int_V \varphi + \int_D d\varphi . \tag{33}$$

Using Eq. (31), we have

$$\int_D d\varphi = -\int_D d(i_k A) = \int_V d^p\sigma \, \Phi - \int_U d^p\sigma \, \Phi \tag{34}$$

and, hence,

$$\int_U d^p\sigma [\nu\sqrt{\det m} + \Phi] = \int_V \varphi + \int_V d^p\sigma \, \Phi \leq \int_V d^p\sigma [\nu\sqrt{\det m} + \Phi], \tag{35}$$

where the inequality follows from Eq. (29). We conclude that

$$E(U) \leq E(V), \tag{36}$$

where E is the energy, and, hence, that U, initially defined as a subset of the contact set of a generalized calibration, is also a subset of a minimal world space. Since U is arbitrary, we conclude that the entire calibrated manifold of a generalized calibration is a minimal world space, as claimed. We remark that if $d i_k A = 0$, then the above bound of the generalized calibration reduces to the bound for calibrated manifolds saturated by minimal surfaces.

Before proceeding to the examples we should discuss one potential difficulty with the application of the above ideas. In many cases of interest, static branes are invariant not only under time translations, but also under some number, l, say, of space translations. In fact, in these cases the invariance group is usually the Poincaré group of isometries of $\mathbb{E}^{(1,l)}$, in which case the brane configuration has an interpretation as an l-brane soliton on the p-dimensional world space, i.e., an l-brane ‘‘world-volume soliton.’’ For $l > 0$, the energy density is independent of some directions and the total energy will be proportional to the volume of $\mathbb{E}^l$, which is infinite. This problem can be resolved by periodic identification of the l coordinates. Alternatively, we may minimize the energy per unit l-volume, which we also denote by E. This is

$$E = \int d^{p-l}\sigma [\nu_l \sqrt{\det m} + \Phi_l], \tag{37}$$

where (for static spacetimes) $\nu_l = \sqrt{\det m_l}$, with m_l being the induced metric on $\mathbb{E}^{(1,l)}$, and m is now the induced metric on a section of the world volume with constant $\mathbb{E}^{(1,l)}$ coordinates. Similarly, Φ_l is induced by $i_{k_0} \cdots i_{k_l} A$ where $k_0, \ldots, k_l$ are the translational Killing vector fields of $\mathbb{E}^{(1,l)}$. This is the functional used in [9] to construct the AdS calibration bound. For every configuration B that minimizes the above energy functional, we find a p-brane solution with topology $\mathbb{E}^{(1,l)} \times B$. The considerations of supersymmetry that we have described above for the $l = 0$ case can easily be extended to the $l > 0$ cases. In particular, a generalized calibration satisfies

$$\varphi|_\xi \leq \nu_l \, \mathrm{vol}_\xi , \qquad d\varphi = \mathcal{F}^l, \tag{38}$$

where $\mathcal{F}^l = -d i_{k_0} \cdots i_{k_l} A$.

III. $M2$-BRANES AND GENERALIZED CALIBRATIONS

We shall begin by discussing some examples of ordinary calibrations in nonflat backgrounds. This will allow us to provide an interpretation in terms of calibrations of some recent work on sigma-model lump solitons on the $M2$-brane [8]; specifically, we show that they correspond to calibrated two-surfaces associated to Kähler calibrations.[2] We will then turn to the cases associated with generalized calibrations.

The general setup for lumps on an $M2$-brane is a probe $M2$-brane in a supergravity background of the form $\mathbb{E}^{(1,2)} \times M_8$ with metric

$$ds^2 = ds^2(\mathbb{E}^{(1,2)}) + ds_8^2. \tag{39}$$

We embed the world volume and fix the world-volume reparametrizations such that the $\mathbb{E}^{(1,2)}$ coordinates are identified with the world-volume coordinates. We then consider static world-volume configurations $y^a = y^a(\sigma^1,\sigma^2)\}$, where $\{y^a, a=1,...,8\}$ are coordinates for M_8 and (σ^1,σ^2) are the world-space coordinates. The $M2$-brane energy is then

$$E = \int d^2\sigma \sqrt{\det m}, \tag{40}$$

where m is the metric induced from ds_8^2. We see that, in this case, the redshift factor ν is unity, and Φ vanishes.

The case considered in [8] was $M_8 = \mathbb{E}^4 \times M_4$ with a Kaluza-Klein (KK) monopole metric on M_4. This is a circle bundle over $\mathbb{E}^3$ with the circle degenerating to a point at the ''centers'' of the metric, which are points in $\mathbb{E}^3$. The subcase considered in [8] was the two-center metric, for which there is a privileged direction in $\mathbb{E}^3$ defined (up to a sign) by the line between the two centers. Let n be a unit vector in this direction. The metric is hyper-Kähler, so there are three complex structures I_r $(r=1,2,3)$. In particular, the linear combination

$$J = n^r I_r \tag{41}$$

is a complex structure. The lump configuration of the $M2$-brane discussed in [8] is associated with the two-form

$$\varphi = dx^1 \wedge dx^2 + \omega_J, \tag{42}$$

where ω_J is the Kähler form of J with respect to the hyper-Kähler metric on M_4. This is a Kähler form on $\mathbb{E}^2 \times M_4$ and satisfies the inequality (29). It is therefore a calibration. The calibration bound is

$$E \geqslant |\varphi|,$$

which is saturated by the lump configurations. These are desingularized intersections of the $M2$-brane probe with an $M2$-brane wrapped on the finite area holomorphic two-cycle of the background. This cycle can be described as follows. Let θ be the S^1 coordinate for the S^1 fiber of M_4 and choose the centers in $\mathbb{E}^3$ of the M_4 metric to be at $u = \pm L$ on the u axis. Then the two-cycle is $\{\theta, u;\ -L \leqslant u \leqslant L\}$.

To find the energy of the lump soliton one must subtract the vacuum energy of the $M2$-brane probe. To see how this may be done we note that the energy of the lump is independent of its size, which is arbitrary. A zero-size lump corresponds to a point intersection of the $M2$-brane probe with the wrapped $M2$-brane. The two $M2$-branes can then be moved apart, without changing the energy, so that they ''overlap'' rather than intersect. We then have two separate $M2$-branes, each of which must be a calibrated surface. This is evident for the probe $M2$-brane, which is now an infinite planar membrane in $\mathbb{E}^4$; it is calibrated by the two-form $dx^1 \wedge dx^2$, and its energy is the infinite vacuum energy. The energy of the lump soliton is therefore equal to that of the wrapped $M2$-brane. The energy is finite since the volume of this surface is finite. The surface itself, described above, is calibrated by the Kähler two-form ω_J.

Various generalizations of this construction are possible. A multicenter metric has various finite area holomorphic two-cycles around which an $M2$-brane may be wrapped. Intersections with another infinite planar $M2$-brane are again, when desingularized, lump solitons, although there is now more than one type of lump. Another generalization arises from the fact that the eight-metric ds_8^2 could be any eight-dimensional Ricci-flat metric, although it must have reduced holonomy if the background is to preserve any supersymmetry. The possible reduced holonomy groups are SU(2), SU(3), SU(4), G_2, Spin(7), and Sp(2). For $M2$-brane probes in these backgrounds the only relevant calibrations are of degree 2, and the supersymmetric backgrounds admitting a degree-2 calibration are those for which the holonomy group is SU(2), SU(3), SU(4), or Sp(2). All these cases admit Kähler calibrations; the calibrated surfaces are holomorphic two-dimensional subspaces.

So far, we have not needed the full power of *generalized* calibrations. These arise by placing $M2$-branes in a $D=11$ supergravity background with nonvanishing four-form field strength. An example is provided by an $M2$-brane probe placed parallel to a $M2$-brane background. The relevant calibrations that describe the supersymmetric dynamics of the probe are Hermitian of degree 2. The metric m and the electrostatic potential Φ of Eq. (1) are induced via the map $x^i(\sigma) = \sigma^i$, $y^a = y^a(\sigma)$, from the metric $ds^2 \equiv g_{IJ} dY^I dY^J$ and the three-form $\mathcal{F} \equiv -d i_k A$, respectively, where

$$ds^2 = H^{-2/3}[(dx^1)^2 + (dx^2)^2] + H^{1/3}\delta_{ab} dy^a dy^b,$$

$$\mathcal{F} = dH^{-} \wedge dx^1 \wedge dx^2, \tag{43}$$

and H is the familiar harmonic function of the multi $M2$-brane supergravity solution. The redshift factor is $\nu = H^{-1/3}$.

To define the Hermitian-generalized calibration relevant to this case it suffices to give the Hermitian form of the complex structure on the ten-space. This is

$$\omega_J = H^{-1} dx^1 \wedge dx^2 + \omega_K, \tag{44}$$

[2] Solutions to the Cayley calibration equations found in [18] were characterized there as ''octonionic lumps.''

where ω_K is a (constant) Kähler two-form for $\mathbb{E}^8$. Note that $\mathcal{F}=d\omega_J$, as required for a generalized calibration. In addition, ω_J is the two-form obtained from the almost-complex structure by raising an index with the rescaled metric

$$d\tilde{s}^2=\nu ds^2=H^{-1}[(dx^1)^2+(dx^2)^2]+\delta_{ab}dy^a dy^b. \quad (45)$$

The almost complex structure is constant and hence integrable. It then follows from Wirtinger's inequality that ω_J is a generalized calibration. The calibrated surfaces are holomophic curves in the $M2$-brane background. Note that the metric $\tilde{m}$ induced on world space by the rescaled metric $d\tilde{s}^2$ is such that $\sqrt{\det\tilde{m}}=\nu\sqrt{\det m}$.

There are many holomorphic curves in the $M2$-brane background. For embeddings of the form $x^i=\sigma^i$, $y^a=y^a(\sigma)$, as above, they are roughly characterized by the number of "active" transverse scalars. From the bulk perspective, every pair of active scalars is interpreted as the world-volume coordinates of another $M2$-brane, so the bulk interpretation of the solution with $2k$ active scalars is that of $k+1$ $M2$-branes intersecting, or overlapping (not necessarily orthogonally), on a 0-brane soliton, all in the given $M2$-brane background. This interpretation suggests that the proportion of the 32 supersymmetries preserved by the probe $M2$-brane is $2^{-(k+1)}$, in agreement with the proportion derived using the contact set $SU(k+1)/S(U(1)\times U(k))$ of the calibration (note that for these examples the presence of the background does not affect the count of supersymmetries).

IV. *M*5-BRANES AND GENERALIZED CALIBRATIONS

Lumps are also relevant to static $M5$-brane configurations. For this we place an $M5$-brane in a background of the form $\mathbb{E}^{(1,5)}\times M_5$ where the M_5 metric is Ricci flat and, for preservation of supersymmetry, has holonomy in SU(2). It follows that $M_5=\mathbb{E}\times M_4$, with a direct product metric such that the M_4 metric has holonomy in SU(2). The SU(2) Kähler calibration of degree 2 discussed above is again applicable; the interpretation is as $M5$-branes intersecting on a three-brane in a KK-monopole background. However, there are now other, higher-degree, calibrations to consider. One such case is a degree-4 Kähler calibration, which we now describe. Let (x^0,x^i) $(i=1,2,3,4,5)$ be Cartesian coordinates for $\mathbb{E}^{(1,5)}$, and let y^a $(a=1,...,4)$ be coordinates on M_4. Consider the manifold $\mathbb{E}^4\times M_4$ where $\mathbb{E}^4$ is the subspace of $\mathbb{E}^{(1,5)}$ with constant x^0 and x^5. This is a Kähler manifold with Kähler two-form

$$\omega=dx^1\wedge dx^2+dx^3\wedge dx^4+\omega_J, \quad (46)$$

where ω_J is defined as for the case of the $M2$-brane. We fix the world-volume diffeomorphisms, and partially specify the embedding of the $M5$-brane in spacetime, by choosing $x^0=t$ and identifying x^i with the world-space coordinates σ^i. In addition, we choose the four functions y^a to be independent of both the time and one of the world-space coordinates. This means that the $M5$-brane has topology $\mathbb{E}^{(1,1)}\times B$ where B is a four-surface parametrized by the other four world-space coordinates. The four functions y^a define a minimal embedding of B in M_4 associated to the calibration four-form

$$\varphi=\frac{1}{2}\,\omega\wedge\omega. \quad (47)$$

If we take $M_4=K_3$, then the $M5$-brane is wrapped on K_3 and can be identified with the heterotic string [19]. We have now seen that this wrapping can be described by saying that B is a minimal four-surface in K_3 calibrated by an SU(2) Kähler calibration of degree 4.

Examples of *generalized* calibrations can be found by considering an $M5$-brane probe in an $M5$-brane supergravity background, for which the metric and seven-form field strength are, respectively,

$$ds^2=H^{-1/3}ds^2(\mathbb{E}^{(1,5)})+H^{2/3}ds^2(\mathbb{E}^5),$$
$$F=-dH^{-1}\wedge dx^0\wedge dx^1\wedge\cdots\wedge dx^5, \quad (48)$$

where H is a harmonic function on $\mathbb{E}^5$, which we choose to be

$$H=1+\mu/r^3$$

for positive constant μ. We assume here that the self-dual three-form field strength on the $M5$-brane vanishes; otherwise the theory of generalized calibrations elaborated here would be inapplicable. The Bianchi identity for this field strength then implies that the pull-back to the world volume of the four-form F must also vanish. This condition is not satisfied for five active scalars, so we consider only those cases with four or fewer active scalars; for a multicenter $M5$-brane solution further restrictions would be needed. There are still many more cases to consider than there were for the $M2$-brane. These include "Hermitian," "special almost symplectic" (SAS) and "exceptional" calibrations, generalizing the corresponding calibrations that have been investigated in the "near-horizon" AdS backgrounds [9]. The calibration bounds depends on the degree of the calibration and the number of active scalars. We are therefore led to an organization in which we first specify the number of active scalars and then consider the various degrees for the calibration form. In what follows $\{x^1,\ldots,x^{5-l}\}$ will denote the world-volume coordinates of the $M5$-brane which are transverse to the l-brane world-volume soliton and $\{y^1,\ldots,y^n\}$, $n\leqslant 5$, will denote the transverse scalars. As in the $M2$-brane case, it is convenient to consider a rescaled metric. Taking this rescaling into account, we have

$$d\tilde{s}^2=\nu_l^{1/(5-l)}[H^{-1/3}ds^2(\mathbb{E}^{(5-l)})+H^{2/3}ds^2(\mathbb{E}^n)],$$
$$\mathcal{F}^l=dH^{-1}\wedge dx^1\wedge\cdots dx^{5-l}, \quad (49)$$

where $\nu_l=H^{-(l+1)/3}$. The static solutions will be $M5$-branes with topology $\mathbb{E}^{(1,l)}\times B$, where B is a calibrated surface of dimension $p-l$. We now consider the various subcases in turn.

A. Hermitian $M5$-brane calibrations

There are four Hermitian calibrations relevant to the $M5$-brane which are best distinguished by the group that rotates the complex planes in their contact set. In all cases, the calibrated submanifold B is a holomorphic submanifold of the $M5$-brane background.

SU(2) Hermitian calibrations. These are degree-2 calibrations with two transverse scalars. The relevant four-dimensional metric and form field strength are

$$d\tilde{s}^2 = H^{-1}[(dx^1)^2+(dx^2)^2]+(dy^1)^2+(dy^2)^2,$$
$$\mathcal{F}^3 = dH^{-1}\wedge dx^1\wedge dx^2, \tag{50}$$

respectively. The calibration two-form is

$$\varphi = H^{-1}dx^1\wedge dx^2 + dy^1\wedge dy^2, \tag{51}$$

which is the Hermitian form of a complex structure. The complex coordinates are $\{z^1=y^1+iy^2,\ z^2=x^1+ix^2\}$, and the calibrated submanifold is given by the vanishing locus of a holomorphic function $f(z^1,z^2)$. The ''Killing'' spinors obey the conditions

$$\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\epsilon=\epsilon,$$
$$\Gamma_1\Gamma_2\epsilon=\Gamma_6\Gamma_7\epsilon, \tag{52}$$

where the gamma matrices are in an orthonormal basis. We conclude that the solution preserves $\frac{1}{4}$ of bulk supersymmetry.[3] The bulk interpretation of the solutions is that of two $M5$-branes intersecting [at SU(2) angles] on a three-brane, with one of them parallel to the background.

SU(3) Hermitian calibrations. There are two cases to consider. The first is a degree-2 calibration with four transverse scalars. The relevant metric and form field strength are

$$d\tilde{s}^2 = H^{-1}[(dx^1)^2+(dx^2)^2]+(dy^1)^2+(dy^2)^2+(dy^3)^2 + (dy^4)^2,$$
$$\mathcal{F}^3 = dH^{-1}\wedge dx^1\wedge dx^2, \tag{53}$$

respectively. The calibration two-form is

$$\varphi = H^{-1}dx^1\wedge dx^2 + dy^1\wedge dy^2 + dy^3\wedge dy^4, \tag{54}$$

which is again the Hermitian form of a complex structure. The holomorphic coordinates are $\{z^1=y^1+iy^2,\ z^2=y^3+iy^4,\ z^3=x^1+ix^2\}$, and the calibrated surfaces can be described as the vanishing locus of two holomorphic functions $f^1(z^1,z^2,z^3), f^2(z^1,z^2,z^3)$. The Killing spinors satisfy the conditions

$$\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5=\epsilon,$$
$$\Gamma_1\Gamma_2\epsilon=\Gamma_6\Gamma_7\epsilon,$$
$$\Gamma_1\Gamma_2\epsilon=\Gamma_8\Gamma_9\epsilon, \tag{55}$$

and so the solution preserves $\frac{1}{8}$ of supersymmetry. The bulk interpretation of the configuration is as intersecting $M5$-branes at SU(3) angles for which the corresponding orthogonal intersection is

$$\begin{array}{ll} M5 & 0,1,2,3,4,5,*,*,*,*, \\ M5 & 0,*,*,3,4,5,6,7,*,*, \\ M5 & 0,*,*,3,4,5,*,*,8,9. \end{array} \tag{56}$$

The other SU(3) Hermitian calibration is a degree-4 calibration with two transverse scalars. The relevant metric and form field strength are

$$d\tilde{s}^2 = H^{-1/2}[(dx^1)^2+(dx^2)^2+(dx^3)^2+(dx^4)^2] + H^{1/2}[(dy^1)^2+(dy^2)^2],$$
$$\mathcal{F}^1 = dH^{-1}\wedge dx^1\wedge dx^2\wedge dx^3\wedge dx^4, \tag{57}$$

respectively. To describe the calibration form, we introduce the Hermitian form

$$\omega = H^{-1/2}[dx^1\wedge dx^2 + dx^3\wedge dx^4] + H^{1/2}dy^1\wedge dy^2; \tag{58}$$

the complex structure can be found by raising indices with the metric $d^2\tilde{s}$. The calibration form is $\varphi=\frac{1}{2}\omega\wedge\omega$. In particular,

$$\varphi = H^{-1}dx^1\wedge dx^2\wedge dx^3\wedge dx^4 + dx^1\wedge dx^2\wedge dy^1\wedge dy^2 + dx^3\wedge dx^4\wedge dy^1\wedge dy^2. \tag{59}$$

The holomorphic coordinates are $\{z^1=y^1+iy^2,\ z^2=y^3+iy^4,\ z^3=x^1+ix^2\}$, and the calibrated surfaces can be described as the vanishing locus of a holomorphic function $f(z^1,z^2,z^3)$. The Killing spinors satisfy the conditions

$$\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\epsilon=\epsilon,$$
$$\Gamma_1\Gamma_2\epsilon=\Gamma_6\Gamma_7\epsilon,$$
$$\Gamma_3\Gamma_4\epsilon=\Gamma_6\Gamma_7\epsilon, \tag{60}$$

and so the solution preserves $\frac{1}{8}$ of bulk supersymmetry. The bulk interpretation of the configuration is as intersecting $M5$-branes at SU(3) angles for which the corresponding orthogonal intersection is

$$\begin{array}{ll} M5 & 0,1,2,3,4,5,*,*, \\ M5 & 0,*,*3,4,5,6,7, \\ M5 & 0,1,2,*,*5,6,7. \end{array} \tag{61}$$

[3] The first of these conditions can be interpreted as due either to the background or to the probe, so the examples being considered are those for which the background does not impose additional constraints.

SU(4) Hermitian calibration. The calibration of interest from the $M5$-brane perspective is the degree-4 calibration with four transverse scalars. The relevant metric and form field strength are

$$d\tilde{s}^2 = H^{-1/2}[(dx^1)^2+(dx^2)^2+(dx^3)^2+(dx^4)^2] + H^{1/2}[(dy^1)^2+(dy^2)^2+(dy^3)^2+(dy^4)^2],$$
$$\mathcal{F}^1 = dH^{-1}\wedge dx^1\wedge dx^2\wedge dx^3\wedge dx^4, \tag{62}$$

respectively. To describe the calibration form we introduce the Hermitian form

$$\omega = H^{-1/2}[dx^1\wedge dx^2+dx^3\wedge dx^4] + H^{1/2}[dy^1\wedge dy^2+dy^3\wedge dy^4]. \tag{63}$$

Then the calibration four-form is $\varphi=\frac{1}{2}\omega\wedge\omega$, i.e.,

$$\varphi = H^{-1}dx^1\wedge dx^2\wedge dx^3\wedge dx^4+[dx^1\wedge dx^2\wedge dy^1\wedge dy^2 + dx^1\wedge dx^2\wedge dy^3\wedge dy^4+dx^3\wedge dx^4\wedge dy^1\wedge dy^2 + dx^3\wedge dx^4\wedge dy^3\wedge dy^4]+H dy^1\wedge dy^2\wedge dy^3\wedge dy^4. \tag{64}$$

The complex coordinates are $\{z^1=y^1+iy^2,\ z^2=y^3+iy^4,\ z^3=x^1+ix^2,\ z^4=x^3+ix^4\}$, and the calibrated manifolds can be described as the vanishing locus of the two holomorphic functions $f^1(z^1,z^2,z^3,z^4), f^2(z^1,z^2,z^3,z^4)$. The conditions on the Killing spinors are

$$\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\epsilon=\epsilon,$$
$$\Gamma_1\Gamma_2\epsilon=\Gamma_6\Gamma_7\epsilon,$$
$$\Gamma_3\Gamma_4\epsilon=\Gamma_6\Gamma_7\epsilon,$$
$$\Gamma_6\Gamma_7\epsilon=\Gamma_8\Gamma_9\epsilon, \tag{65}$$

and so the solution preserves $\frac{1}{16}$ of bulk supersymmetry.

In all of these cases it is clear that $\mathcal{F}^l=d\varphi$ as required for consistency. In addition, in all the above cases an alternative interpretation of the solutions is as $M5$-branes with topology $\mathbb{E}^{(1,l)}\times C$, for either $l=1$ or $l=3$, where C is a $(5-l)$-dimensional holomorphic cycle.

B. SAS calibrations

The special almost symplectic calibrations of $AdS_7\times S^4$, which is the near-horizon geometry of the $M5$-brane, have a natural generalization in the full $M5$-brane background. The SAS calibrations associated with an $M5$-brane probe placed parallel to an $M5$-brane supergravity background have degrees 2, 3, or 4 (a calibration of degree 5 would have to involve the two-form tensor field and is therefore excluded). The degree-2 SAS calibration is identical to the degree-2 Hermitian calibration explained in the previous section. It therefore remains to describe the SAS calibrations with degrees 3 and 4. For SAS calibrations the number of transverse scalars is the same as the degree of the calibration. So the degree-3 SAS calibration is a three-dimensional submanifold in a six-dimensional manifold and the degree-4 calibration is a four-dimensional submanifold in a eight-dimensional manifold.

For the degree-3 SAS calibration, the relevant metric and form field strength are

$$d\tilde{s}^2 = H^{-2/3}[(dx^1)^2+(dx^2)^2+(dx^3)^2]+H^{1/3}[(dy^1)^2 + (dy^2)^2+(dy^3)^2],$$
$$\mathcal{F}^2 = dH^{-1}\wedge dx^1\wedge dx^2\wedge dx^3, \tag{66}$$

respectively.

To proceed we introduce the orthonormal frame

$$e^p = H^{-1/3}dx^p,\quad 1\leqslant p\leqslant 3,$$
$$e^{3+p}=H^{1/6}dy^p,\quad 1\leqslant p\leqslant 3. \tag{67}$$

The almost symplectic form is defined as

$$\omega \equiv \sum_{p=1}^{3} e^p\wedge e^{3+p} = H^{-1/6}\sum_{i=1}^{3} dx^i\wedge dy^i. \tag{68}$$

Using the almost complex structure associated with this almost symplectic form, the calibration form is found to be

$$\varphi \equiv \mathrm{Re}(e^1+ie^4)\wedge\cdots\wedge(e^3+ie^6) = H^{-1}dx^1\wedge dx^2\wedge dx^3 - dx^1\wedge dy^2\wedge dy^3 - dy^1\wedge dx^2\wedge dy^3 - dy^1\wedge dy^2\wedge dx^3. \tag{69}$$

Repeating the same arguments as in the AdS case in [8], the calibrated submanifolds are seen to be determined by a single real function $f(x^1,x^2,x^3)$ such that

$$y^i=\frac{\partial}{\partial x^i}f(x^1,x^2,x^3),$$
$$H^{-1}\delta^{mn}\partial_m\partial_n f=\det(\partial_i\partial_j f). \tag{70}$$

The Killing spinors of such solutions satisfy the conditions

$$\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\epsilon=\epsilon,$$
$$\Gamma_0\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_7\epsilon=-\epsilon,$$
$$\Gamma_0\Gamma_2\Gamma_4\Gamma_5\Gamma_6\Gamma_8\epsilon=\epsilon, \tag{71}$$

and so the configuration preserves $\frac{1}{8}$ of the bulk supersymmetry. The bulk interpretation of the solution is that of three intersecting $M5$-branes, possibly at SU(3) angles, for which the associated orthogonal intersection is

$$M5 \quad 0,1,2,3,4,5,*,*,*,$$
$$M5 \quad 0,*,*,3,4,5,6,7,*,$$
$$M5 \quad 0,*,2,*,4,5,6,*,8. \tag{72}$$

Alternatively, the solution describes an $M5$-brane with topology $\mathbb{E}^{(1,2)}\times B$, where B is a three-dimensional SAS-calibrated submanifold.

For the degree-4 calibration, the relevant metric and form field strength are

$$d\tilde{s}^2 = H^{-1/2}[(dx^1)^2+(dx^2)^2+(dx^3)^2+(dx^4)^2] + H^{1/2}[(dy^1)^2+(dy^2)^2+(dy^3)^2+(dy^4)^2],$$
$$\mathcal{F}^1 = dH^{-1}\wedge dx^1\wedge\cdots\wedge dx^4, \tag{73}$$

respectively. We again introduce the frame

$$e^p = H^{-1/4}dx^p, \quad 1\le p\le 4,$$
$$e^{4+p} = H^{1/4}dy^p, \quad 1\le p\le 4, \tag{74}$$

and the symplectic form

$$\omega \equiv \sum_{p=1}^{4} e^p\wedge e^{4+p} = \sum_{i=1}^{3} dx^i\wedge dy^i. \tag{75}$$

Using again the almost complex structure associated with ω, the calibration form is found to be

$$\begin{aligned}\varphi &= \mathrm{Re}(e^1+ie^4)\wedge\cdots\wedge(e^4+ie^8)\\ &= H^{-1}dx^1\wedge dx^2\wedge dx^3\wedge dx^4 - dx^1\wedge dx^2\wedge dy^3\wedge dy^4 - dx^1\wedge dy^2\wedge dx^3\wedge dy^4 - dx^1\wedge dy^2\wedge dy^3\wedge dx^4 - dy^1\wedge dy^2\wedge dx^3\wedge dx^4\\ &\quad - dy^1\wedge dx^2\wedge dy^3\wedge dx^4 - dy^1\wedge dx^2\wedge dx^3\wedge dy^4 + H dy^1\wedge dy^2\wedge dy^3\wedge dy^4.\end{aligned} \tag{76}$$

As in the previous case, the solutions are determined by a single real function $f(x^1,x^2,x^3,x^4)$ such that

$$y^i = \frac{\partial}{\partial x^i} f(x^1,x^2,x^3,x^4),$$
$$H^{-1}\delta^{mn}\partial_m\partial_n f = \sum_m \det{}_{m|m}(\partial_i\partial_j f), \tag{77}$$

where $\det_{m|m}$ denotes the determinant of a matrix with the mth row and column omitted. The Killing spinors of these solutions obey the conditions

$$\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\epsilon = \epsilon,$$
$$\Gamma_1\Gamma_2\epsilon = -\Gamma_6\Gamma_7\epsilon,$$
$$\Gamma_1\Gamma_3\epsilon = -\Gamma_6\Gamma_8\epsilon,$$
$$\Gamma_1\Gamma_4\epsilon = -\Gamma_6\Gamma_9\epsilon, \tag{78}$$

and so they preserve $\frac{1}{16}$ of bulk supersymmetry. The bulk interpretation of the solutions is as four intersecting $M5$-branes, possibly at SU(4) angles, for which the corresponding orthogonal intersection is

$$M5 \quad 0,1,2,3,4,5,*,*,*,*,$$
$$M5 \quad 0,*,*,3,4,5,6,7,*,*,$$
$$M5 \quad 0,*,2,*,4,5,6,*,8,*,$$
$$M5 \quad 0,*,2,3,*,5,6,*,*,9. \tag{79}$$

Alternatively, the solution describes an $M5$-brane with topology $\mathbb{E}^{(1,1)}\times B$, where B is a four-dimensional SAS submanifold.

C. Exceptional calibrations

There are three exceptional calibrations to consider as follows. The Cayley calibration which is a degree-4 calibration with four transverse scalars. The relevant eight-dimensional metric and form field strength are

$$d\tilde{s}^2 = H^{-1/2}[(dx^1)^2+(dx^2)^2+(dx^3)^2+(dx^4)^2] + H^{1/2}[(dy^1)^2+(dy^2)^2+(dy^3)^2+(dy^4)^2],$$
$$\mathcal{F}^1 = dH^{-1}\wedge dx^1\wedge\cdots\wedge dx^4, \tag{80}$$

respectively. The calibration four-form can be constructed from the metric and a Spin(7) invariant self-dual four-form on $\mathbb{E}^8$, viz.,

$$\begin{aligned}\varphi &= H^{-1}dx^1\wedge dx^2\wedge dx^3\wedge dx^4 + [dx^3\wedge dx^4\wedge dy^3\wedge dy^4 + dx^2\wedge dx^4\wedge dy^2\wedge dy^4 + dx^2\wedge dx^3\wedge dy^2\wedge dy^3 + dx^1\wedge dx^3\wedge dy^2\wedge dy^4\\ &\quad + dx^1\wedge dx^2\wedge dy^1\wedge dy^2 + dx^1\wedge dx^3\wedge dy^1\wedge dy^3 + dx^1\wedge dx^4\wedge dy^1\wedge dy^4 + dx^2\wedge dx^4\wedge dy^1\wedge dy^3 - dx^1\wedge dx^2\wedge dy^3\wedge dy^4\\ &\quad - dx^1\wedge dx^4\wedge dy^2\wedge dy^3 - dx^3\wedge dx^4\wedge dy^1\wedge dy^2 - dx^2\wedge dx^3\wedge dy^1\wedge dy^4] + H dy^1\wedge dy^2\wedge dy^3\wedge dy^4.\end{aligned} \tag{81}$$

The equations which the transverse scalars satisfy are

$$H^{-1}(\partial_1 Y-\partial_2 Yi-\partial_3 Yj-\partial_4 Yk)$$
$$=\partial_2 Y\times\partial_3 Y\times\partial_4 Y+\partial_1 Y\times\partial_3 Y\times\partial_4 Yi-\partial_1 Y\times\partial_2 Y$$
$$\times\partial_4 Yj+\partial_1 Y\times\partial_2 Y\times\partial_3 Yk, \qquad (82)$$

together with the second order auxiliary condition

$$\mathrm{Im}[(\partial_1 Y\times\partial_2 Y-\partial_3 Y\times\partial_4 Y)i+(\partial_1 Y\times\partial_3 Y+\partial_2 Y\times\partial_4 Y)j$$
$$+(\partial_1 Y\times\partial_4 Y-\partial_2 Y\times\partial_3 Y)k]=0, \qquad (83)$$

where $Y=\underline{y}^1+i\underline{y}^2+jy^3+ky^4$, $a\times b\times c=\frac{1}{2}(a\bar{b}c-c\bar{b}a)$, $a\times b=-\frac{1}{2}(\bar{a}b-\bar{b}a)$, and i,j,k are the imaginary unit quaternions. The Killing spinors satisfy the conditions

$$\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\epsilon=\epsilon,$$
$$\Gamma_3\Gamma_4=-\Gamma_8\Gamma_9\epsilon,$$
$$\Gamma_2\Gamma_3\epsilon=-\Gamma_7\Gamma_8\epsilon,$$
$$\Gamma_1\Gamma_2\epsilon=-\Gamma_6\Gamma_7\epsilon,$$
$$\Gamma_1\Gamma_4\epsilon=\Gamma_7\Gamma_8\epsilon, \qquad (84)$$

and so the solution preserves $\frac{1}{32}$ of bulk supersymmetry.

The associative calibration is a degree-3 calibration with four transverse scalars. The relevant seven-dimensional metric and form field strength are

$$d\tilde{s}^2=H^{-2/3}[(dx^1)^2+(dx^2)^2+(dx^3)^2]$$
$$+H^{1/3}[(dy^1)^2+(dy^2)^2+(dy^3)^2+(dy^4)^2],$$
$$\mathcal{F}^2=dH^{-1}\wedge dx^1\wedge dx^2\wedge dx^3, \qquad (85)$$

respectively. The calibration three-form is constructed from the above metric and the structure constants of octonions as follows:

$$\varphi=H^{-1}dx^1\wedge dx^2\wedge dx^3+[dx^1\wedge dy^1\wedge dy^2+dx^3\wedge dy^1\wedge dy^4$$
$$+dx^2\wedge dy^2\wedge dy^4+dx^2\wedge dy^1\wedge dy^3-dx^1\wedge dy^3\wedge dy^4$$
$$-dx^3\wedge dy^2\wedge dy^3]. \qquad (86)$$

The equations which the transverse scalars satisfy are

$$-H^{-1}(\partial_1 Yi+\partial_2 Yj+\partial_3 Yk)=\partial_1 Y\times\partial_2 Y\times\partial_3 Y, \qquad (87)$$

where Y is defined as in the Cayley case. The conditions on the Killing spinor are

$$\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\epsilon=\epsilon,$$
$$\Gamma_2\Gamma_3\epsilon=-\Gamma_8\Gamma_9\epsilon,$$
$$\Gamma_1\Gamma_3\epsilon=-\Gamma_7\Gamma_9\epsilon,$$
$$\Gamma_1\Gamma_3\epsilon=-\Gamma_6\Gamma_8\epsilon, \qquad (88)$$

and so the configuration preserves $\frac{1}{16}$ of the supersymmetry.

The coassociative calibration is a degree-4 calibration with three transverse scalars. The relevant seven-dimensional metric and form field strength are

$$d\tilde{s}^2=H^{-1/2}[(dx^1)^2+(dx^2)^2+(dx^3)^2+(dx^4)^2]$$
$$+H^{1/2}[(dy^1)^2+(dy^2)^2+(dy^3)^2],$$
$$\mathcal{F}^1=dH^{-1}\wedge dx^1\wedge\cdots\wedge dx^4, \qquad (89)$$

respectively. The calibration four-form is constructed from the above metric and the dual form of the structure constants of octonions in $\mathbb{E}^7$ as

$$\varphi=H^{-1}dx^1\wedge dx^2\wedge dx^3\wedge dx^4+[dx^3\wedge dx^4\wedge dy^2\wedge dy^3$$
$$+dx^2\wedge dx^4\wedge dy^1\wedge dy^3+dx^2\wedge dx^3\wedge dy^1\wedge dy^2$$
$$+dx^1\wedge dx^3\wedge dy^1\wedge dy^3-dx^1\wedge dx^2\wedge dy^2\wedge dy^3$$
$$-dx^1\wedge dx^4\wedge dy^1\wedge dy^2]. \qquad (90)$$

The transverse scalars of this calibration satisfy the equations

$$-H^{-1}(\partial y^1 i+\partial y^2 j+\partial y^3 k)=\partial y^1\times\partial y^2\times\partial y^3, \qquad (91)$$

where $\partial=\partial_1+i\partial_2+j\partial_3+k\partial_4$; $\partial_i=\partial/\partial x^i$. The conditions on the Killing spinors are

$$\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\epsilon=\epsilon,$$
$$\Gamma_3\Gamma_4\epsilon=-\Gamma_7\Gamma_8\epsilon,$$
$$\Gamma_1\Gamma_2\epsilon=\Gamma_7\Gamma_8\epsilon,$$
$$\Gamma_2\Gamma_4\epsilon=-\Gamma_6\Gamma_8\epsilon, \qquad (92)$$

and so the solutions preserve $\frac{1}{16}$ of bulk supersymmetry. In all the above three types of calibration, $\mathcal{F}^l=d\varphi$ as required for consistency.

V. CONCLUSIONS

We have shown that generalized calibrations arise naturally in the context of supersymmetric configurations of p-brane actions. In particular, we have derived the generalized calibration bound from the p-brane supersymmetry algebra. We have also presented several examples of such calibrations in M-brane backgrounds. These examples by no means exhaust the possibilities. An obvious generalization would be to consider $\frac{1}{4}$ supersymmetric, intersecting-brane, backgrounds determined by two harmonic functions. More generally still, one can take any supersymmetric intersecting brane configuration in flat spacetime and replace one of the participating branes by its corresponding supergravity solution. One then expects to recover the full intersecting brane configuration as a generalized calibration of a probe brane in this background. Our examples are exclusively *M* theoretic, but string theory examples can easily be found and many are simply related to those discussed here by some duality chain.

The main limitation of calibrations is that the theory is inapplicable in those cases for which there are ''active'' world volume vectors or tensors, as can happen for string theory *D*-branes and the *M*5-brane. Here again, however, many such cases are related to calibrations via duality chains, so this limitation is not as severe as might be thought. We should point out that calibrations were discussed in the *D*-brane context in [20]. There is also the possibility that a reformulation of *D*-branes and the *M*5-brane in which world-volume vectors and tensors are induced from an extended spacetime, as recently proposed [21], might allow a direct extension of the theory of calibrations to these cases too.

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